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Beyond continuous mathematics and traditional scientific analysis: Understanding and mining Wolfram's *A New Kind of Science*

J.J McDowell*, Andrei Popa

Department of Psychology, Emory University, Atlanta, GA 30322, United States

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ABSTRACT

In *A New Kind of Science*, Stephen Wolfram recommends abandoning traditional scientific analysis and the continuous mathematical description that it affords in favor of the study of simple rules. He focuses on a machine known as a cellular automaton as the prototype generator of complex phenomena such as those we see in nature. The simplest cellular automaton consists of a row of cells, each existing in one of two states. The states of the cells are updated from moment to moment by simple rules. Wolfram shows that these machines and their many variations can generate a host of outcomes ranging from very simple to extremely complex. He argues that among these outcomes representations of all the phenomena in the universe will be found, including presumably the behavior of organisms. The output of cellular automata can be mapped to behavior by considering, for example, one of the states of a cell to represent the emission of a behavior. For some cellular automaton rules, these mappings generate cumulative records and inter-response time distributions that are similar to those produced by live organisms. In addition, at least one cellular automaton generates the Herrnstein hyperbola as an emergent outcome. These results suggest that Wolfram's program and its mainstream version, which is known as complexity theory, is worth pursuing as a possible means of understanding and accounting for the behavior of organisms.

True science is distinctively the study of useless things.
Charles Sanders Peirce (Peirce, 1955/1896, p. 48).

A New Kind of Science (Wolfram, 2002) is a registered trademark of Stephen Wolfram, LLC, scientist–entrepreneur, author of influential published articles in particle physics by age 17, Ph.D. in theoretical physics from the California Institute of Technology at age 20, Professor at the California Institute of Technology at age 21, MacArthur fellow at age 22, now-wealthy inventor of *Mathematica*, the gold standard in symbolic calculation by computer, jealous and litigious guardian of his intellectual property rights (e.g., Giles, 2002; Wolfram, 2002, copyright page), unabashed self-promoter (e.g., Wolfram, 2008b), tireless solicitor of media attention (e.g., Wolfram, 2008c), and possibly diagnosable narcissist (e.g., Giles, 2002; Hayes, 2002; see also Acknowledgements). Wolfram self-published *A New Kind of Science* in 2002 after a lengthy exile (Wolfram, 2002), during which he completed this, his *magnum opus*. It is a large format book weighing about 6 lb and covering 1280 pages. There are 840 pages of main text and 440 pages of small-print notes, each section containing about 250,000 words. Approximately 1000 illustrations are included in the text and notes. This massive tome is now available online (Wolfram, 2008a).

Reviews of *A New Kind of Science* have been mixed (e.g., *inter plures alius*, Bailey, 2002; Cybenko, 2002; Gray, 2003; Giles, 2002; Hayes, 2002; Mitchell, 2002; Weinberg, 2002). Nearly every reviewer remarks on Wolfram's breathtaking hubris and unabashed egocentricity. Reviewers have also criticized Wolfram's often inadequate citation of the literature, his failure to accurately attribute findings to others, his sometimes embarrassingly inadequate background in fields in which he is not expert, and his conjunctive-ridden, repetitive prose that is desperately in need of editing. On the positive side, Wolfram's program is usually seen as ambitious and provocative. Steven Weinberg (2002), Nobel laureate in physics for his work on the Standard Model of particle physics, sums up the consensus view: "I don't think that [Wolfram's] book comes close to meeting his goals or justifying his claims, but if it is a failure it is an interesting one." Later in his review Weinberg offers a more pointed assessment: "...as far as I can tell, there is not one real-world complex phenomenon that has been convincingly explained by Wolfram's computer experiments." And, regarding Wolfram's influence, Weinberg notes: "Since [the early 1990s], none of his work has had much of an impact on the research of other scientists, aside from Wolfram's employees." It is possible to find apologists for *A New Kind of Science* (e.g., Boguta, 2004; Cawley, 2007), but they too tend to be on Wolfram's payroll.

In spite of the chilly reception of *A New Kind of Science* and its serious flaws, this book is filled with interesting ideas, fascinating methods, and alluring pathways, some of which hint of exotic and

* Corresponding author.
E-mail address: psyjjmd@emory.edu (J.J McDowell).

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fruitful destinations. The best way to approach *A New Kind of Science* is as a scientist's personal notebook, written in private (which it was) and not meant for publication, but containing bits and pieces of ideas and methods, some better developed than others, some alarmingly uninformed, and some utterly fanciful. Grandiose claims and bad writing can be overlooked in a personal notebook, which is really a font from which material can be recovered at a later date to consider more carefully, develop more fully, and ultimately publish in more conventional forms. Now that Wolfram has published this notebook, however, any interested reader can mine the rich veins of material that it contains.

1. Wolfram's core message

Wolfram's stated purpose in *A New Kind of Science* is to initiate a revolution in science that is comparable in impact to the revolution occasioned by applying mathematics to the description of natural phenomena (Wolfram, 2002, p. 1). This use of mathematics was originally urged by the Pythagoreans around 500 BCE, was later championed by Plato, and first bore fruit in a useful scientific way in the work of Eudoxus of Cnidus who, during the 4th century BCE, applied the well-developed geometry of his day to the motion of celestial bodies (McDowell, 1988; Pedersen and Pihl, 1974). But Wolfram's revolution would set aside the continuous mathematics that has served science for over two millennia, along with the traditional method of scientific analysis that seeks to extract essential variables from, or to generate idealized representations of, natural phenomena. In their place Wolfram proposes to focus on the study of simple rules, rules that may be implemented as steps in a computer program that when executed causes a numeric or graphical representation of natural phenomena to emerge in a fully complex form. Successful abstract rules, which are those that generate accurate representations of the complex phenomena of interest, can be viewed as completely accounting for the phenomena to which they give rise. No higher-order mathematical descriptions of simplified or idealized versions are necessary.

Importantly, Wolfram's starting point is not the phenomena of interest, but the simple rules and programs that might generate representations of them. This is the sense in which he wishes to discard traditional scientific analysis along with continuous mathematical description. The answers are to be found in the study of simple rules, not in the analysis of natural phenomena into simplified or idealized forms that permit mathematical description. The correct rules will produce the phenomena in all their complexity straightaway. Wolfram (2002) is confident that "all the wonders of our universe can. . .be captured by simple rules (p. 846)".

Wolfram's hoped-for revolution represents an interestingly constructivist view of science and, perhaps, of truth. In effect, to understand a phenomenon one must be able to create it by means of rules that can be implemented in computer software. It follows that to be considered true, a scientific account must be capable of creating a representation of the phenomenon of interest. This is reminiscent of a view advanced nearly 300 years ago by the Neapolitan scholar, Giambattista Vico in a completely different context, namely, that of human history and social and political theory. In an unnerving instance of synchronicity (Jung, 1991/1952) the title of Vico's (1984, 1744) *magnum opus*, also self-published, was *Scienza Nuova* (*The New Science*). As Wolfram did nearly three hundred years later, Vico rejected the dominant hypothetico-deductive approach of Descartes, a forerunner of modern science, including what Vico considered the Cartesian conceit of mathematical description. In their place Vico advanced his principle of *verum factum* (Vico, 1998/1709), which asserts that what is true is what is made. "The criterion or rule of the true is that it has been made (Vico, 1998/1709)." Stated another way: "to know [something] is to. . .bring [it] about (Morrison, 1978, p. 584)." Moreover, according to Morrison (1978), "what Vico seeks [in human science] is the universal and eternal *form* of the real and changing *particulars* of history: a philosophy (science) of the historical" (pp. 591–592, italics and parenthetical expression in the original). For Wolfram, the Vichian universal and eternal forms are simple abstract rules.

To develop a human science, Vico felt it necessary to reject the approach and methods of the existing science of inanimate objects. Three hundred years later, Wolfram seeks to abandon the same scientific approach and methods *even in* the study of inanimate objects, thus making a clean sweep of Vico's rejection of traditional science. Wolfram seeks a unified science of everything, a science based on Vichian universal and eternal forms, realized as sets of simple abstract rules that give rise to all the particulars of the universe including, surely, the behavior of organisms. Notice that Wolfram's full realization of the Vichian program is the antithesis of behavior analytic tradition, and especially of the program of quantitative behavior analysis. Both behavior analysis and the Vico–Wolfram program seek to apply the same kind of science to all phenomena, but for behavior analysis it is the traditional science of analysis and mathematical description, whereas for the Vico–Wolfram program it is the science of universal and eternal forms that give rise to the particulars of the world.

2. The universal and eternal forms: cellular automata

The centerpiece of *A New Kind of Science* is the cellular automaton, which is a machine invented in the 1940s by Stanislaw Ulam and John von Neumann at the Los Alamos National Laboratory. The simplest cellular automaton is a 1-dimensional, 2-state, nearest-neighbor machine that evolves from a set of initial conditions according to simple rules. A cellular automaton consists of rows of squares, or cells, (the one dimension) that may be either black or white (1 or 0, the two states). The first row of cells constitutes the automaton's initial condition. A second row is produced from the first by means of rules that determine the state of each cell in the second row based on its state in the first row and the states of its left and right neighbors in that row (the nearest neighbors). For example, a white cell in the first row may have two white neighbors, two black neighbors, a white left and black right neighbor, or a black left and white right neighbor. Depending on its neighbor condition, the white cell is required to remain white in the second row or it is required to turn black. A black cell in the first row has the same four possible neighbor conditions. Hence the rules of a cellular automaton consist of the eight possible neighbor conditions of a cell, along with the outcome of each one, which is the state of that cell in the next row. From the second row of cells, a third row is generated using the same rules, and from the third row a fourth row is generated, and so on. Because there are 8 possible neighbor conditions and 2 possible outcomes for each one, there are 2^8 or 256 sets of rules for 1-dimensional, 2-state, nearest neighbor cellular automata.

The three-column table in Fig. 1 lists the eight possible neighbor conditions for a cell (labeled "Target") in a cellular automaton, with 0 representing white and 1 representing black. Notice that reading the digits in each row, left to right, from the top row to the bottom row, i.e., 000, 001, 010, etc., gives the sequence in binary notation of the decimal integers 0–7. This table is the same for every 1-dimensional, 2-state, nearest neighbor cellular automaton. The "Next" column to the right of the table lists the state of the target cell in the next row, given the neighbor condition specified in the same line of the main table. For example, for the cellular automaton defined in Fig. 1, a white cell with two white neighbors will remain white in the next row, a white cell with a white left and a black right neighbor will turn black, and so on. The "Next" column uniquely defines each cellular automaton. Taking the bits in this column from the bottom to the top, which in Fig. 1 gives 00010110,

Left	Target	Right	Next
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

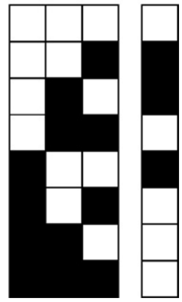

Fig. 1. Definition of a Rule 22 cellular automaton. The eight possible neighbor conditions are listed in the 3-column table on the left, and in graphical form in the 8×3 black/white grid of cells to the right (0 = white; 1 = black). The outcome of each neighbor condition is listed in the "Next" column, and is also shown graphically in the black/white column of cells on the extreme right. The numerical designation of the cellular automaton comes from the "Next" column which, when read from bottom to top is 00010110. This is the binary representation of the decimal integer 22.

yields a binary number that identifies the cellular automaton. In this case the binary number corresponds to the decimal integer 22, and hence this automaton is often referred to as a Rule 22 cellular automaton, or simply as Rule 22. The black/white 8×3 grid on the right side of Fig. 1 is a graphic way to specify the possible neighbor conditions of a cellular automaton, and the 8×1 grid to its right is a graphic "Next" column. Software that can be used to generate cellular automata is discussed in the Appendix B.

The time evolution of a Rule 22 cellular automaton is shown in Fig. 2 for 885 time steps (rows), with the initial condition (first row) consisting of the bit string, 101101, centered and surrounded by 0s. This bit string is represented in the figure by a sequence of black (1) and white (0) cells. Each row in the image contains 256 cells. The rows are wrapped such that the right neighbor of the last cell in a row is the first cell in that row, and the left neighbor of the first cell is the last cell. The first 295 time steps are shown in the left panel of the figure; the second and third 295 times steps are shown in the center and right panels. The image in Fig. 2 is "zoomed out", making the individual cells indistinguishable. What is strikingly apparent, however, is that the simple rule specified in Fig. 1 generates a host of irregular low- and high-order structures and patterns. The lowest-order structures are individual white triangles of various sizes. Higher-order structures consist of assemblies of triangles, or have borders defined by assemblies of triangles. For example, a circle appears centered near the bottom of the top-half of the middle panel of Fig. 2, with a diameter equal to about 1/2 the width of the panel (best detected when the figure is held at arm's length). Other higher-order structures are readily apparent in the image.

The image in Fig. 2 exhibits mirror symmetry about its midline because of the cellular automaton's initial condition, but once the edges of the panel are reached, the structures centered on the midline, and those located entirely on either side of it, do not repeat during the 885 time steps shown in the figure. These structures and patterns are emergent properties of Rule 22 and cannot be predicted from the rule table in Fig. 1. For this reason, the outcome shown in Fig. 2 is said to be computationally irreducible. A computationally irreducible outcome is one that can be obtained only by carrying out the computations specified by a rule, and not by any analytic shortcut. Wolfram believes that many natural phenomena are computationally irreducible, and hence cannot be described in any way other than by specifying the rules that generate them.

In addition to generating complicated patterns, cellular automata are capable of carrying out specific calculations and computations. The top panel of Fig. 3 shows the time evolution of Rule 132, which is defined in Table 1. Rule 132 is able to classify a number, n, as even or odd when the number is input as an initial condition consisting of n consecutive black cells surrounded by at least one white cell. As shown in the figure, when n is even, the automaton stops producing black cells after $n/2$ time steps, but when n is odd it produces a single column of black cells indefinitely after $(n-1)/2$ time steps. The bottom panel of Fig. 3 shows the time evolution of

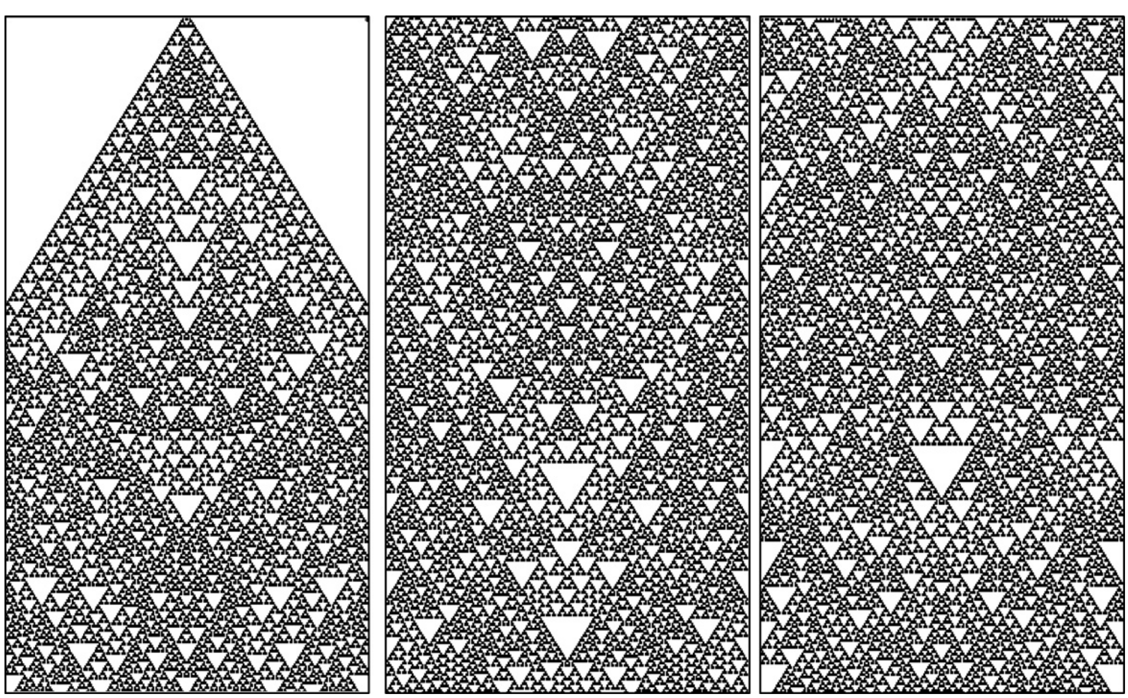

Fig. 2. Time evolution of a Rule 22 cellular automaton. The evolution continues from the left to the center to the right panel for a total of 885 time steps. Many low-and high-order structures emerge from Rule 22. Notice, for example, the "smiling" triangle spanning the vertical midline near the top of the lower half of the left panel. This structure is made up of many smaller triangles and its smile spans the center 1/3 of the image.

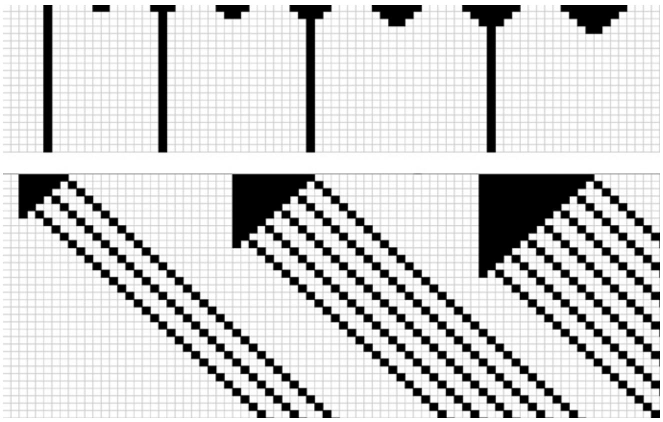

Fig. 3. Top: time evolution of Rule 132 given $n = \{1, 2, 3, \ldots, 8\}$ adjacent black cells as initial conditions. Rule 132 classifies n as even or odd. Bottom: time evolution of Rule 152 given $n = \{6, 10, 14\}$ adjacent black cells as initial conditions. Rule 152 calculates the quantity, $(n + n \bmod 2)/2$, and outputs the result as diagonal black lines, or shooters, the number of which gives the result of the calculation.

Table 1
Cellular automaton rules.

Neighbor condition			Rule			
Left	Target	Right	30	110	132	152
0	0	0	0	0	0	0
0	0	1	1	1	0	0
0	1	0	1	1	1	0
0	1	1	1	1	0	1
1	0	0	1	0	0	1
1	0	1	0	1	0	0
1	1	0	0	1	0	0
1	1	1	0	0	1	1

Note: Rule columns show the state of the target cell in the next row given the corresponding neighbor condition.

Rule 152, which is also defined in Table 1. Rule 152 calculates the quantity, $(n + n \bmod 2)/2$, when n is input as an initial condition consisting of n consecutive black cells surrounded by at least one white cell. For even n, this calculation amounts to dividing by 2, as shown in the bottom panel of Fig. 3 for $n = 6$, 10, and 14. The number of black diagonal shooters, which consist of at least one cell after $n + n \bmod 2$ time steps, and continue indefinitely, gives the result of the calculation. With appropriate initial conditions, various cellular automata are capable of carrying out a vast array of categorizations, calculations, and computations, including logical operations such as AND, NOT, and XOR.

A cellular automaton of special interest is Rule 110, which is defined in Table 1. The time evolution of this cellular automaton, having an initial condition (row) of 256 squares colored black with a probability of 0.5, is shown in Fig. 4 for 885 time steps. The first, second, and third segments (295 times steps each) of the evolution are shown in the left, middle, and right panels of the figure. The rows are wrapped as described earlier. The image in Fig. 4 shows the emergence of well-defined structures that move through time (down the page) and collide and interact with each other, sometimes producing new structures. After an initial burst of debris at the top of the left panel, specific structures coalesce, such as a vertical column of white right triangles with a black edge, various thick and thin, single and multiple, black-line structures that travel to the left or right, and various coiled structures that move at different angles through space (across the image) and time. The output of this cellular automaton shows in a striking way that simple rules can produce complex outcomes. But Rule 110 turns out to be even more complex than Fig. 4 would suggest. Matthew Cook (2004) proved that Rule 110 is capable of universal computation. That is, given appropriate initial conditions, Rule 110 can perform any computation whatsoever, just as can any laptop or desktop computer. But unlike these computers, the operation of Rule 110 is completely determined by the rules listed in the eight lines of Table 1.

The lesson of cellular automata, especially Rule 110, is that apparent complexity, such as we observe in the natural world, can

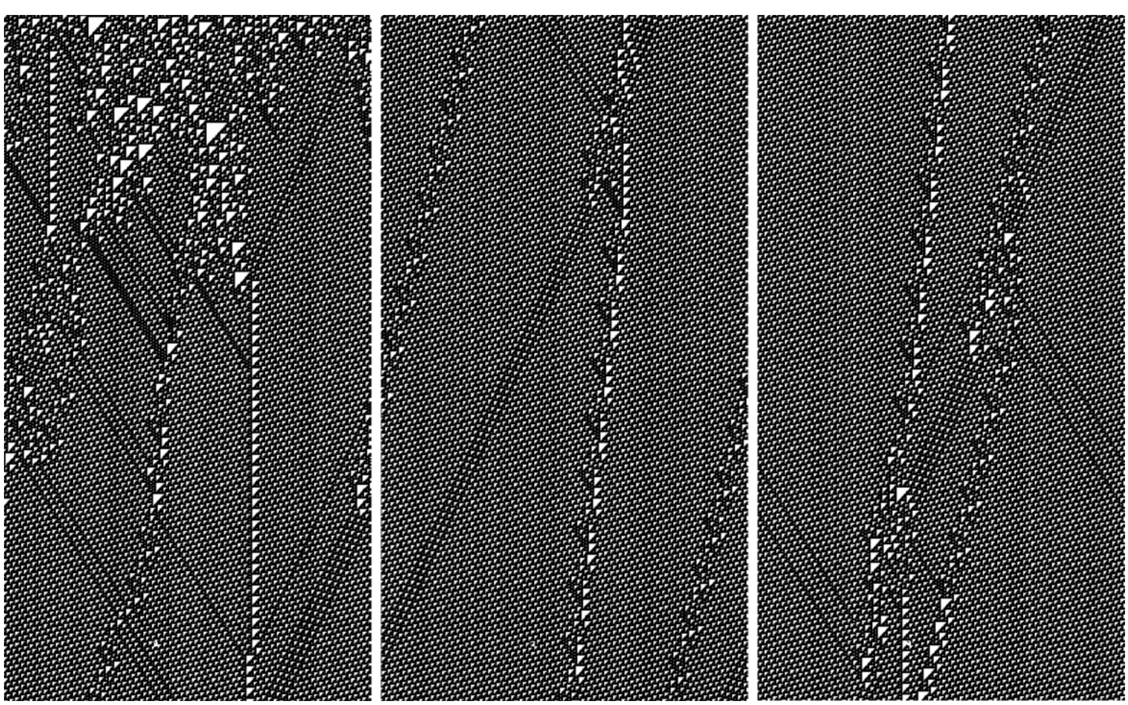

Fig. 4. Time evolution of Rule 110 with an initial condition consisting of random black cells. The time evolution continues from the left to the center to the right panel for a total of 885 time steps. The debris from the initial explosion at the top of the left panel rapidly condenses into a number of well-defined structures that move through time (down the page), and collide and interact with each other.

be the emergent outcome of simple rules. Moreover, if the complexity is computationally irreducible, then only the simple rules constitute a full explanation of it. Beginning with an image like that in Fig. 4, we might attempt to classify the different structures, study their lifetimes and velocities (direction and speed), and attempt to catalogue and describe their interactions with other structures. This is the approach of traditional science; it begins with the phenomena. There is no doubt that this can lead to interesting results (and in fact has done so, e.g., Wolfram, 2002, pp. 291ff), but these are just bits and pieces of the whole picture. The whole picture, which is to say, the fundamental truth of the phenomena, is contained in the simple rules that produce them. Wolfram believes that natural phenomena are the product of simple rules, are in most cases computationally irreducible, and must therefore be accounted for by finding the simple rules that give rise to them. Hence the necessity of the revolution, the new kind of science.

3. The iceberg below sea level

One-dimensional, 2-state, nearest neighbor cellular automata are the simplest forms of these machines and consequently are sometimes referred to as elementary cellular automata. Of course one can construct cellular automata with more dimensions and states, and that have more neighbors contributing to their rules. The number of a cellular automaton's states is usually denoted by k. The number of neighbors on each side of a cell that contribute to determining its state at the next time step is usually referred to as the cellular automaton's radius and is denoted by r. For the 1-dimensional cellular automata we have considered so far, $k=2$ and $r=1$. For 1-dimensional cellular automata, the number of possible neighbor conditions, which is the number of lines that must be in its rule table, is k^{2r+1}. For each of these neighbor conditions, there are k possible outcomes or states of the cell at the next time step, which means there are $k^{k^{2r+1}}$ possible cellular automata. As we have seen, for $k=2$ and $r=1$ there are 256 possible cellular automata. If we increase the number of states to three, {0, 1, 2}, which could be represented by white, gray, and black colors, but retain $r=1$, there are 7,625,597,484,987 possible cellular automata. Evidently, the universe of cellular automata expands explosively as k and r are increased.

Wolfram's principled focus on rules often leads him to recommend exhaustive searches of a universe of cellular automata to learn about the outcomes they generate. Because of computational irreducibility, searching a universe of cellular automata means running them, and Wolfram is not reluctant to take on this task. For example, on p. 833 (Wolfram, 2002) he reports searching all 7.6 trillion 1-dimensional, 3-state, nearest neighbor cellular automata for those that exactly double the width of an initial condition consisting of n consecutive black cells. He found that 0.0000000561% of the cellular automata, or 4277, carried out this calculation. This is a good illustration of how one goes about doing Wolfram's new kind of science, and what it is like to do it. The scientist has a phenomenon in mind ("doubling" in this example), and then searches a vast universe of simple rules for the potentially very rare result that corresponds to the phenomenon.

This is also a good illustration of how Wolfram's program is actually closer to pure mathematics than to what is commonly understood as science. His program is to study the outcomes of simple rules. Whether these outcomes correspond to phenomena in the natural world is a separate question, although Wolfram's intuition is that they will include representations of all phenomena in the universe. In this vein, Wolfram might agree with Peirce's sentiment, noted at the beginning of this article, that science is the study of useless things, useless in the sense that they need not be applicable immediately to specific problems. Hence the sheer study of abstract rules is a worthwhile project for Wolfram. There is no doubt that the value of this perspective is supported by the history of science and mathematics. Selecting a circle, as Eudoxus did (perhaps at Plato's urging), from the various objects of abstract geometry that were invented and studied for their own sake, and using it to describe celestial motion is an obvious example. An awe-inspiring modern example is the appearance of properties of prime numbers in the energy levels of atoms observed in modern experiments. Prime numbers were known as mathematical objects by the Pythagoreans before 400 BCE. The connection between prime numbers and atomic energy states was made via Hermitian matrices (operators) and Bernhard Riemann's zeta function, both of which are purely mathematical objects that were developed in the 19th century (Derbyshire, 2003). These mathematical objects, which remained "unapplied" over the course of 2000 or so years, turned out to correspond to some of the most fundamental properties of the natural world. No doubt Wolfram's background in theoretical physics, a field in which there is a close interplay between mathematics and natural phenomena, makes him comfortable with his focus on simple rules.

The universe of cellular automata is even larger than that suggested by increasing the dimensionality, number of states, and ranges of these machines. For example, cellular automata can be constructed such that the state of a cell depends not merely on the states of its neighbors at the previous time step, but on some function of those states. If this function is the sum (or average) of the states of the cell and its neighbors at the previous time step, then the cellular automaton is referred to as totalistic in Wolfram's terminology, and as outer totalistic in the general literature (e.g., Gray, 2003). In addition to being an entirely different class of rules, totalistic cellular automata reduce the universe of their corresponding elementary versions, making it more tractable. For example, 1-dimensional, 3-state, nearest neighbor totalistic cellular automata have only 7 lines in their rules table (because the states of three cells with values 0, 1, or 2 have possible sums from 0 to 6), instead of the 27 lines in the rules table of the non-totalistic versions of this automaton. Consequently, there are only 2187 totalistic 3-state automata, a universe greatly reduced from the approximately 7.6 trillion non-totalistic 3-state automata.

Another interesting variation is the continuous cellular automaton. The states of cells in this automaton vary continuously from 0 to 1, and correspond graphically to continuous levels of gray from white (0) to black (1). The state of a cell at a time step is some function of the average states of the cell and its nearest neighbors at the previous time step. Obviously, the universe of continuous cellular automata defined in this way is extremely large, limited only by the number of functions that are applied to the average.

Wolfram also discusses iterated maps, which are interesting because they consist of rules (like the well-known logistic map) that generate a number from a seed according to some function, and then generate a second number from the first according to the same function, and so on. Wolfram converts the iterated decimal numbers to binary form and then, oddly, enters the digit sequences of these binary numbers in successive rows of black (1) and white (0) cells, with the digit sequence of each number iterated by the map occupying one row. The patterns that emerge in the resulting image then are examined just as for any cellular automaton. Wolfram discusses in detail many additional variations of cellular automata, including mobile cellular automata, Turing machines that are implemented on grids of cells, substitution and sequential substitution systems, tag and cyclic tag systems, register machines, and the all-important causal networks, all important because Wolfram believes that among them a universe rule will be found, a simple rule that will explain everything we see in the universe. The elementary cellular automata and their many variations constitute rich veins of material that can be mined by interested scientists for many years.

4. Applying cellular automata to behavior

Cellular automata have been used to represent and understand ballistic collisions in particle mechanics, turbulence in fluid flow, the flow of automobile traffic, and the Belousov–Zhabotinsky reaction, which is a non-linear chemical oscillation. Using cellular automata to represent natural phenomena such as these entails mapping features of the output of the machines onto physical properties of the phenomena. In the case of behavior, one possible mapping is to consider each column of the output of a cellular automaton as a series of time ticks (which it is) during which a discrete behavior, such as a lever press, may occur. Some pattern of cells, the simplest being a single black or white cell, can represent an occurrence of the behavior. Successive columns in the output of a cellular automaton may be taken to represent the continuation of an experimental session, or additional sessions, or sessions with different subjects. Given this mapping it is possible to create cumulative records, prepare inter-response time (IRT) distributions, and calculate average response rates from the outputs of cellular automata.

Wolfram's approach would be to pursue a mapping like this one to see where it goes. Let's try Wolfram's approach. Fig. 5 shows mappings of Rule 22 (top row of images, rule defined in Fig. 1) and Rule 30 (bottom row of images, rule defined in Table 1) to cumulative records, and to IRT distributions in the form of log survivor plots. Log survivor plots show the log of the proportion of IRTs of length *t* or greater as a function of *t* (Kulubekova and McDowell, 2008; Shull et al., 2001). Put another way, the plot shows the proportion of IRTs in a sample that is still surviving *t* units of time after the last response. The log transform of the proportion of surviving IRTs makes patterns in IRT distributions more apparent, especially for samples with predominantly short IRTs.

The cellular automaton output images in Fig. 5 are 145-row by 120-column sections of the larger 1000-row by 256-column images that were used to prepare the cumulative records and log survivor plots shown in the figure. The generally smaller features and apparent greater density of black cells in the Rule 30 image are emergent outcomes of that rule, not an artifact of level of zoom. The initial condition for each automaton was a row of cells colored black with a probability of 0.5. Each black cell in a column was taken to represent a response, and cumulative records and IRT distributions were prepared for a number of columns, the results for one of which is shown in the figure for each rule.

The cumulative records for both rules show roughly steady response rates, but also exhibit occasional irregularities, including pauses of varying lengths and brief periods of response acceleration and deceleration. Cumulative records like these are typical of the behavior of many species responding on variable interval (VI) schedules of reinforcement. Overall, Rule 22 generated an average response rate of about 33 responses per 100 time ticks, and Rule 30 generated an average response rate of about 50 responses per 100 time ticks. This difference in overall rate is consistent with the apparently greater density of black cells in the Rule 30 image. For both Rules 22 and 30, cumulative records constructed from different columns had similar although not identical features.

As shown in Fig. 5, the log survivor plot for Rule 22 was irregular, suggesting that it might be a stochastic variation of a straight line. A linear log survivor plot corresponds to an exponential IRT survivor function, which means that response emission can be characterized as a Poisson process. But this was not the case for Rule 22 inasmuch as the specific irregular form of the log survivor plot shown in the figure appeared to be representative of plots generated from other columns. Indeed, it is known that live organisms typically are not Poisson responders. For example, Davison (2004) studied pigeons' responding on concurrent VI VI schedules and reported irregular log survivor plots that were similar to the plot shown for Rule 22 (and also for Rule 30). Davison demonstrated that Gaussian log-normal analyses could be used to characterize the various twists and turns of these irregular plots. While the log survivor plot for Rule 30 was also irregular, its form varied considerably from column to column.

As the examples in Fig. 5 show, columnar mappings from Rules 22 and 30 to behavior produced cumulative records and log survivor plots that were similar to those produced by live organisms responding on VI schedules. According to Wolfram's classification scheme, Rules 22 and 30 are examples of Class 3 cellular automata, which exhibit irregular, non-patterned outcomes. Wolfram's Class 4 cellular automata include Rule 110 (Fig. 4), and exhibit well-defined structures that move through time and space and interact with each other. Although Class 4 cellular automata generate more complex, and in many ways more interesting outcomes, it seems unlikely that they would yield good models of behavior, at least with the columnar mapping used here. With this mapping, responding

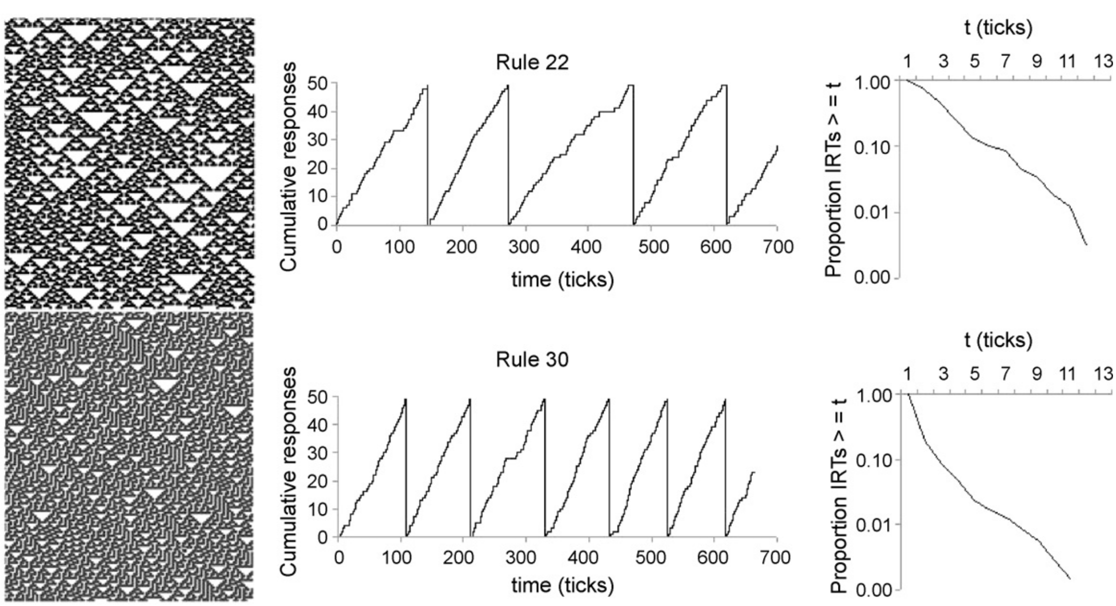

Fig. 5. Image of a section of cellular automaton output, a cumulative record, and a log IRT survivor plot for a Rule 22 (top row) and a Rule 30 (bottom row) cellular automaton. The cumulative records and log survivor plots were constructed from a column of the cellular automaton output where each black cell was taken to represent a response.

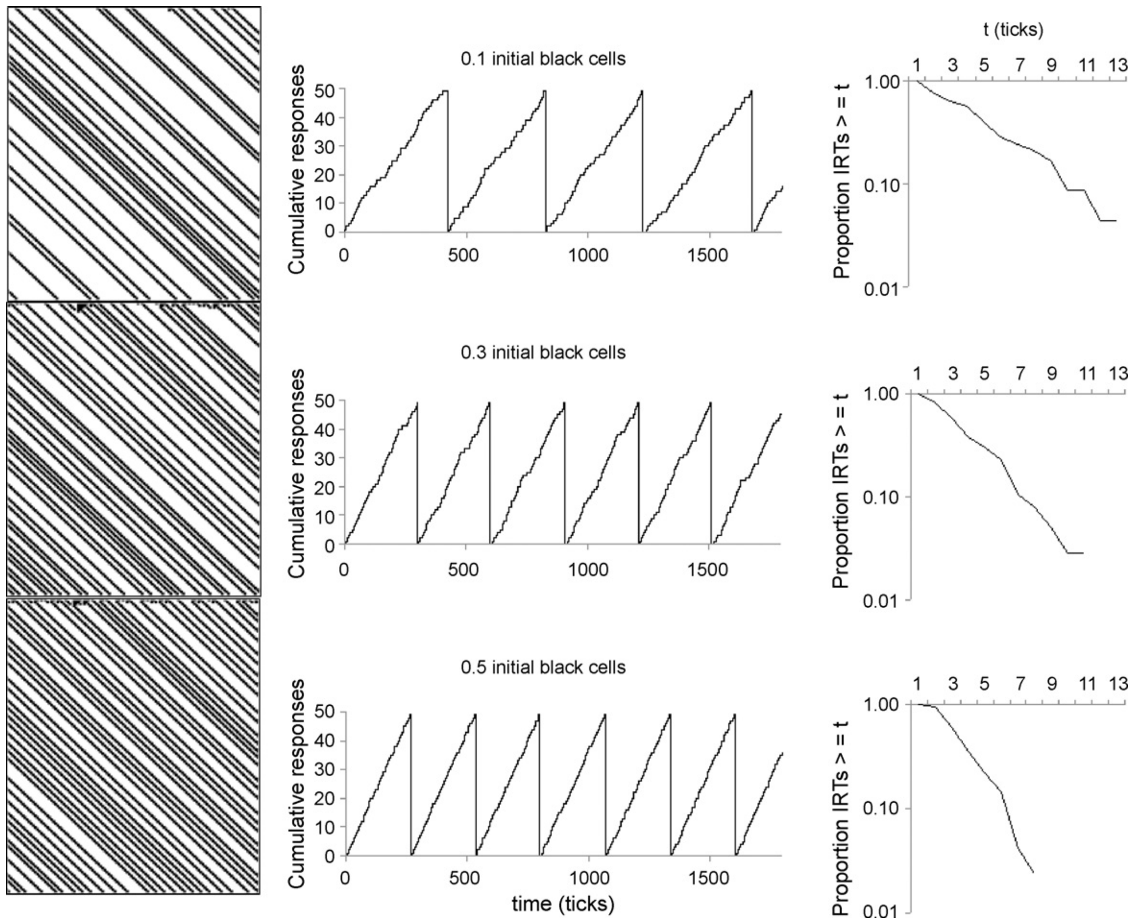

Fig. 6. Output images of Rule 152 cellular automata, and columnar mappings to cumulative records and log survivor plots for initial conditions consisting of 10% (top panels), 30% (middle panels), and 50% (bottom panels) black cells.

would go through highly repetitive epochs generated, for example, by the background pattern in Fig. 4, with occasional bursts of irregular responding that would be generated when a column passed through a structure or a region of interaction between structures.

To complete Wolfram's classification scheme, Class 1 cellular automata have outcomes with a uniform final state, such as all black cells, or a constant checkerboard pattern, and Class 2 cellular automata have relatively simple repetitive structures. Rule 152 (bottom panel of Fig. 3) is an example of a Class 2 cellular automaton. Some cellular automaton rules produce outcomes that fall into different classes depending on their initial conditions. For example, Rule 22 produces a Class 3 outcome (Figs. 2 and 5) with random initial conditions, but when the initial condition is a single black cell it produces a Class 2 repetitive fractal pattern known as a Sierpiński gasket. The outputs of elementary cellular automata become more complex as one moves from Class 1 to 4.

Obviously the representations of behavior generated by columnar mappings of Rules 22 and 30 are incomplete. For example, how is input to the system, such as reinforcement, represented? One way that this might be accomplished is to use a property of the initial condition of the automaton, such as the proportion of black cells in the first row, to represent this input variable. Unfortunately, for Rules 22 and 30, the columnar frequency of black cells rapidly converges on the same value (which is different for the two rules) regardless of the initial condition of the automaton, and hence it is an emergent property of each rule. This means that different response rates would have to be represented by different rules, perhaps all from Class 3, rather than by some property of a single rule that can take on multiple values.

Although we have not exhaustively searched the universe of elementary cellular automata, computer experiments in our laboratory suggest that very few generate different columnar frequencies of black cells when given different proportions of black cells as the initial condition. However, one rule that does have this property is Rule 152, which is a Class 2 cellular automaton that generates a much simpler output than any Class 3 or 4 machine. We have seen an example of this output when the initial condition consisted of blocks of black cells (bottom panel of Fig. 3). But given random initial conditions, Rule 152 produces outcomes like those shown in the images in Fig. 6. An image of the output of Rule 152 where 10% of the cells in the first row were randomly selected and assigned a state of 1 (black), is shown in the top left panel of the figure, along with a cumulative record and a log IRT survivor plot built from one of the columns in that image. For the middle panels, 30% of the cells in the first row of the cellular automaton were randomly selected and assigned a state of 1, and for the bottom panels, 50% of the cells in the first row were randomly selected and assigned a state of 1. For mappings to cumulative records and log survivor plots, a response was defined as a block of three cells with consecutive states, 100 (black–white–white). This definition was used because black cells in columns of Rule 152 are never separated by fewer than two white cells. Defining a response as a 3-cell block consisting of one black followed by two white cells permits responses to occur without pauses, and hence allows for continuous bouts of responding. As is clear from Fig. 6, after the first few time steps, the overall average response rates generated by Rule 152 varied directly with the proportion of black cells in the initial condition of the automaton. Results essentially identical to these are obtained if a response is defined as a 2-cell block consisting of a black followed by a white cell.

The images in Fig. 6 are 145-row by 120-column sections of larger 1000-row by 256-column images that were used to prepare the cumulative records and log survivor plots in the figure. As was the case for Rules 22 and 30, the cumulative records showed fairly constant rates of responding given each initial condition, with various irregularities such as might be seen in the behavior of a live organism responding on VI schedules. The log survivor plots had irregular forms. Much larger samples of responding (>1000 time steps) would be required to fully characterize these forms. Interestingly, for a specific initial condition, each column in the output of Rule 152 generates the same cumulative record, but shifted to the right a number of time steps equal to the number of columns between the records being compared (cf. a linear shift register). Hence, one specific pattern of an initial condition (i.e., a set of specific cells colored black) produces one sequence of responses, which is the same for every column, only shifted in time. However, different specific patterns of the same initial condition (i.e., different, but the same number of, cells colored black) produce different sequences of responses, and hence different cumulative records and log survivor plots.

The functional relationship between the columnar frequency of black cells and the proportion of black cells in the initial condition of Rule 152 is shown in the top panel of Fig. 7. Outputs from 13 initial conditions were studied, ranging from 2.5% of the first-row cells randomly selected and colored black, to 99% of the first-row cells randomly selected and colored black. In the top panel of the figure, the total number of responses from row 100 to row 1000 of a column from the output is plotted against the obtained initial proportion of black cells. The smooth curve drawn through the points shows the surprising result that the functional relationship is hyperbolic, the very function form known to describe the behavior of live organisms responding on VI schedules. In Herrnstein's (1970) notation,

$$B = \frac{kr}{r + r_e},$$

where B and r are the output and input variables, k is the y-asymptote of the hyperbola, and r_e governs the rapidity with which the hyperbola approaches its asymptote. The hyperbola in Fig. 7 was fitted by the method of least squares, and the residuals from the fit, which are plotted in the bottom panel of Fig. 7, appeared to be random. Interestingly, based on our non-exhaustive search of 1-dimensional, 2-state, nearest neighbor cellular automata, we believe that Rule 152 may be the only elementary cellular automaton that gives the hyperbolic result shown in Fig. 7.

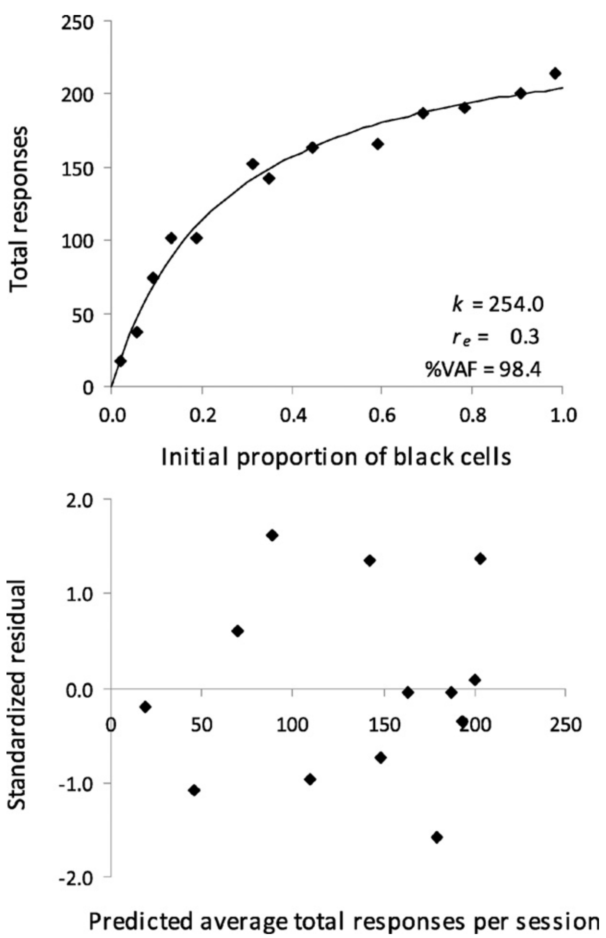

Fig. 7. Top: the Herrnstein hyperbola is an emergent property of Rule 152. The parameters of the best-fitting hyperbola and the percentage of variance it accounts for (%VAF) are listed in the panel. Bottom: standardized residuals for the hyperbolic fit shown in the top panel.

This initial foray into the application of cellular automata to adaptive behavior indicates that the Wolfram program may be worth pursuing in behavior analysis. Needless to say, only the barest beginnings of such an application have been made here. For example, it is obvious that even Rule 152 is not completely satisfactory as a model of responding on single schedules. How do other variables known to affect behavior, such as reinforcer magnitude, find representation in this account? One way that this might be accomplished is by using continuous cellular automata, which were described earlier. The states of cells in these automata are decimals between 0 and 1. A threshold value for the decimal could be selected such that a response would be counted only if the state of a cell exceeded this value. The threshold could represent reinforcer magnitude, or else a more general cost/benefit variable that also takes into account the nature of the response, as does the bias parameter in the power function matching equation. The combination of this threshold value and the initial condition of the continuous cellular automaton might then constitute a workable two-variable input.

There are many additional questions. For example, what might be an approach to concurrent schedules? Staying with the columnar mapping suggested here, the simplest approach would be to use a 1-dimensional, 3-state, nearest neighbor cellular automaton. Two of the three states would represent behavior on each of the two alternatives, and the third would represent behavior on neither alternative, that is, extraneous behavior. These 3-state automata have 27 lines in their rule tables (3^3) and consequently are identified by 27-digit base-3 numbers that define the outcomes of the 27 neighbor conditions. The digit sequences of these base-3 numbers begin at the bottom of the rule table and proceed to the top, just as do the binary numbers that define elementary cellular automata. The base-3 numbers when translated into decimal integers give the decimal codes that are commonly used to identify these automata. Totalistic versions of the 3-state automata, which were discussed earlier, have only 7 lines in their rule tables and are identified by 7-digit base-3 numbers. Based on our essentially random study of a few of the 2187 1-dimensional, 3-state, nearest neighbor totalistic cellular automata, those with decimal codes 177, 912, and 1074 (base-3 outcomes: 0020120, 1020210, and 1110210) appear to be particularly promising candidates for modeling behavior on concurrent schedules. Continuous versions of these automata can also be studied and may permit incorporating more than one input variable into the account.

Two fundamental questions arise from this initial examination of the relevance of Wolfram's *A New Kind of Science* for behavior analysis. These are: what is one ultimately looking for in such an effort, and what would it mean to find it? The answer to the first question is that, ideally, one would like to find a single, reasonably simple cellular automaton, perhaps a 1-dimensional, multi-state, nearest neighbor, continuous cellular automaton, that could generate all known behavioral phenomena, perhaps beginning with the many and varied phenomena that have been extensively documented in behavior-analytic laboratory experiments. This is a tall order, but is surely more modest than Wolfram's quest for a rule that explains the universe. A successful cellular automaton for behavior would constitute a calculation and prediction machine, a quantitative theory, although not one based on continuous mathematics. What would we understand such an automaton to represent, if anything? For example, would we expect such a theory, the main statement of which would consist of a rules table, to correspond to physical reality in some way? Ptolemy's geocentric theory was, and is, an excellent calculation and prediction machine for the motion of many celestial bodies, but we know that it is not an accurate representation of the physical world. As is now well known, the later Copernicus–Kepler heliocentric theory was far superior in this respect. Modern quantum theory is widely regarded as the most complete and accurate calculation and prediction machine ever devised in human history. And yet the manner and extent to which it corresponds to physical reality remains an unresolved issue among many philosophers of science (Leplin, 1984; Van Fraassen, 1980), even if not among practicing theoretical and experimental physicists. Similarly, the meaning and significance of a comprehensive and accurate theory of behavior based on a cellular automaton would likely be a thorny philosophical problem. Ideally, it seems that one would like the elements of such a theory to correspond to the physical world in some way. In the final analysis, however, one might have to be content only with the rules table, just as some philosophers of science (and scientists) argue that in quantum theory one must be content only with the formalisms, such as the Schrödinger wave equation, foregoing any physical interpretation of them (Van Fraassen, 1980).

5. The wider world of complexity theory

Wolfram is the compleat complexity theorist and advocates the most abstract forms of the theory. But complexity theory, which goes by other names including complexity science and complex systems theory, predates Wolfram's *A New Kind of Science*, and has been pursued in a variety of ways other than those Wolfram recommends. Common to all types and applications of complexity theory is the idea that complex phenomena may be the emergent consequence of simple rules of one sort or another.

Straying not too far from the extreme abstraction of the Wolfram program is an approach that entails cellular automata, but first sets up a phenomenon of interest in the form of a cellular automaton output, and then seeks a machine that produces just that outcome. This is different from the Wolfram program because it starts with a specific phenomenon instead of with the simple rules, and consequently may entail methods of finding automata other than by random or exhaustive search. For example, Melanie Mitchell and her colleagues (e.g., Mitchell et al., 1997) pioneered the use of genetic algorithms to evolve cellular automaton rules that generate specific outputs. This same approach could be used with behavior. Time series of responses from live organisms behaving under specific experimental conditions could be used to fill in columns of cells where, say, each response colors a cell black. An appropriately designed genetic algorithm might then be used to search for a cellular automaton that produces a reasonable facsimile of just that outcome. One fault Wolfram likely would find with this approach is that even though it may turn up a cellular automaton that generates the desired output, it might not turn up the simplest such automaton, in which case it would not give the final answer. Another problem is that this approach seems likely to foster the identification of a host of cellular automata, each for a specific purpose, rather than just one automaton, or perhaps just a few, that explain everything. In *A New Kind of Science*, Wolfram repeatedly recommends resisting the urge to expect and therefore search for complicated answers to questions about the natural world, believing it far more likely that the answers to even the most complicated questions are simple.

Moving further away from the abstract Wolfram program, and again predating it, is research in a variety of fields that appeals to simple rules, but not in the form of cellular automata, to explain complex phenomena (e.g., NOVA, July 2007). John Holland, who invented the genetic algorithm, has discussed some of these efforts in general (Holland, 1995, 1998), and technical terms (Holland, 1992). Peter Bentley (2002) has written an engaging account of complexity theory in the biological sciences, and Miller and Page (2007) discuss applications of complexity theory to social systems.

It is worth noting that theoretical approaches consistent with the basic premise of complexity theory are not missing from the behavior-analytic literature. For example, as early as 1966 Shimp proposed a theory of momentary maximizing that is based on a simple local rule, namely, that at any moment an organism emits just the behavior that has the highest likelihood of payoff. Shimp showed that the application of this rule gives rise to the higher-order phenomenon of relative response rate matching. Donahoe and his colleagues (Donahoe et al., 1993; Donahoe and Palmer, 1994) have proposed neural network models of behavior that entail relatively simple rules governing the values of connection weights between artificial neurons. These artificial neural networks have been shown to generate higher-order phenomena, such as the acquisition and extinction of Pavlovian conditional responses (CRs), the faster reacquisition of a previously conditioned CR following its extinction, and the blocking of control of a CR by part of a compound conditional stimulus. As a third example, Catania (2005) based a computational model of instrumental learning on Skinner's idea of the reflex reserve. By implementing local rules that determine how the reserve is incremented and decremented, Catania's model is able to generate a number of phenomena, including the acquisition and extinction of instrumental responding, and characteristic schedule performances of various types. Finally, McDowell (2004) and his colleagues (Kulubekova and McDowell, 2008; McDowell and Caron, 2007; McDowell, Caron, Kulubekova, and Berg, 2008) have implemented Darwinian rules of selection, reproduction, and mutation in a computational theory of selection by consequences. This theory generates behavior on single and concurrent schedules that can be described by the higher-order equations of matching theory.

6. Conclusion

Wolfram (2002) makes the following remarks on ultimate models of the universe:

> Considering the reputation of. . .empirical science, it is remarkable how many significant theories were in fact first constructed on largely aesthetic grounds. Notable examples include Maxwell's equations for electromagnetism (1880s), general relativity (1915), the Dirac equation for relativistic electrons (1928), and QCD [quantum chromodynamics, a theory of the strong nuclear force] (early 1970s). This history makes it seem more plausible that one might be able to come up with an ultimate model. . .on largely aesthetic grounds, rather than mainly by working from detailed experimental observations. (p. 1025)

Many scientists would consider this view parochial in the sense that it applies mainly to theories in physics, and certainly not to theories in biology in any significant way. But Wolfram would argue that this is because the successful aesthetic theories have been mathematical, and mathematical theories work only for the easiest problems, that is, only when the essential properties of natural phenomena can be described in simplified or idealized forms. Note that behavior analysis also has a successful mathematical theory, namely, matching theory, which describes and explains in quantitative detail a large and varied body of data from experiments with many vertebrate species (McDowell, 2005). But consider how this success was achieved—precisely by simplifying the complex phenomena of adaptive behavior, by distilling a number of critical variables, including response and reinforcement rates, from the complicated mix of behavior, organism, and environment. The typical laboratory preparation in behavior analysis is as idealized a form as is the point mass moving in a frictionless medium in classical mechanics.

According to Wolfram, idealized forms and the mathematical theories they afford can only take us so far. To go beyond and deal with fully complex phenomena that are likely to be computationally irreducible, a new kind of science is required. With this new kind of science it may be possible to develop theories of the complex phenomena of biology, including the behavior of organisms, on general or aesthetic grounds "rather than mainly by working from detailed experimental observations". Whether this is a visionary statement or a bit of science fiction remains to be seen. In the meantime, there is no doubt that complexity theory, both of the abstract Wolfram variety and of the more empirically focused mainstream variety, merits careful attention by scientists interested in understanding and accounting for behavior (Staddon and Bueno, 1991).

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Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at doi:10.1016/j.beproc.2009.01.012.

Appendix B

Various Java applets and downloadable programs that generate cellular automata are available on the internet. Wolfram's *Mathematica* software can also be used for this purpose. However, most of these applets and programs are either entertainment oriented or are fairly complicated and entail a non-trivial learning curve. As an alternative, a Microsoft Excel workbook has been made available as a supplement to this article. The workbook contains code that runs all 256 elementary cellular automata. To use the workbook, it must be downloaded and opened in Excel, allowing the workbook's macros to be enabled. A "Read Me" worksheet, which contains instructions on how to operate the software, appears when the workbook is opened. A second worksheet, "Program Notes", describes the most important features of the code that runs the cellular automata, including advice on how to use the code for research. If necessary, Excel help can be consulted to learn how to view the code. A third worksheet, "Template", receives the image of the cellular automaton that is being run.

For a quick start, select the "Template" worksheet and run the "CellularAutomaton" macro (if necessary, consult Excel help to learn how to run a macro). A control panel will appear that includes a "Run Automaton" button. Click the button and a Rule 30 cellular automaton will be drawn on the "Template" worksheet. The cellular automaton's rule number, initial condition, and number of rows and columns may be changed by entering the desired values in the relevant text boxes on the control panel.

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